121 points located at the intersections of the curves. In Fig. 1, the bell-shaped surface

$$W(x,y) = (1 + 9x^2 + 16y^2)^{-1} \quad (14)$$

is difficult to approximate with a polynomial due to the Runge effect (i.e., it has poles nearby in the complex plane). Figure 2 is a part of the cone

$$W(x,y) = 1 - ((x^2 + y^2)/2)^{1/2} \quad (15)$$

The superiority of the surface spline is clearly indicated in both figures.

IV. Modifications

There are several modifications which can be incorporated.

a) Scaling. If the points (x_i, y_i) lie within a long narrow zone, a linear transformation can be and ordinarily should be made to transform the points into a nearly rectangular zone. This is illustrated in Fig. 3.

b) Symmetry. If one or two planes of symmetry or antisymmetry exist, then use can be made of images either to improve accuracy or to reduce the number of simultaneous equations. For example, if $W(x, y)$ is symmetric about $x = 0$, then replace Eq. (9) with

$$W(x, y) = a_0 + a_2 y + \sum_{i=1}^{N} F_i(r_i^2 \ln r_i^2 + \bar{r}_i^2 \ln \bar{r}_i^2) \quad (16)$$

where $\bar{r}_i^2 = (x + x_i)^2 + (y - y_i)^2$. Then set $\sum F_i = \sum y_i F_i = 0$, and $W_j = W(x_j, y_j)$ for the required $N + 2$ equations.

c) Smoothing with elastic springs. Instead of forcing the plate to pass through the N points, apply elastic spring forces to the plate that are proportional to the difference between the desired data point and the smoothed interpolated surface. Mathematically this is the same as Eq. (9) with

$$W_j = a_0 + a_1 x_j + a_2 y_j + \sum_{i=1}^{N} F_i r_{ij}^2 \ln r_{ij}^2 + C_j F_j \quad (17)$$

The coefficients C_j, which have units of length squared, are equal to $16\pi D/K_j$, where D is the plate rigidity and K_j is the spring constant associated with the jth point. For $C_j = 0$ ($K_j = \infty$) we get the original spline, and for $C \to \infty$ ($K \to 0$), the function (11) approaches a least-squares plane fit.

d) Smoothing with distributed loads. If the term $r^2 \ln r^2$ is replaced by $r^2 \ln(r^2 + \varepsilon)$ in both Eqs. (9) and (11), then a new surface is generated which passes through N points. If the loads are computed from Eq. (1), it follows that they are no longer point loads, but are distributed loads. The loads approach point loads as the parameter ε approaches zero. This procedure produces a surface for which all derivatives exist everywhere, and furthermore, it simplifies the coding somewhat.

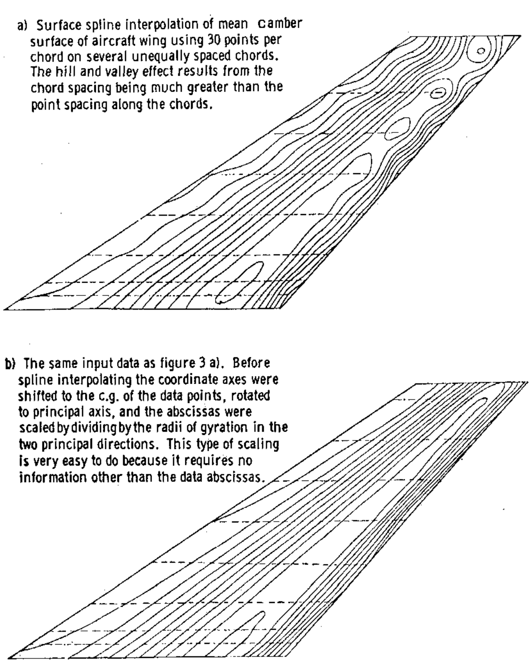

Fig. 3 Effect of scaling data abscissas.

References

[1] Done, G. T. S., "Interpolation of Mode Shapes: A Matrix Scheme Using Two-Way Spline Curves," *Aeronautical Quarterly*, Vol. XVI, Part 4, Nov. 1965, pp. 333–349.

Comment on "Transonic Airfoil Design"

J. Smolderen*
Von Kármán Institute for Fluid Dynamics, Rhode Saint Genèse, Belgium

In their paper,[1] Cahn and Garcia present a method for the solution of the supercritical wing problem, based on a transformation of the polar hodograph coordinates, which reduces this problem to an incompressible potential flow calculation. Using this approach, a mixed elliptic-hyperbolic boundary problem, or even a purely hyperbolic problem could be reduced, in the general case, to a purely elliptic problem, and vice versa.

This result is clearly paradoxical as it contradicts well-known properties of elliptic and hyperbolic equations. For instance, using Cahn and Garcia's technique, it would become possible to construct well behaved solution of a Cauchy problem for an elliptic equation. One simply has to solve the problem in the supersonic region of the compressible hodograph plane, using, say, the method of characteristics, and then use the transformation (7) to obtain a reasonable solution of a Cauchy problem for the Laplace equation in the incompressible hodograph plane. If the Cauchy data used is bounded, but highly oscillatory in nature, then the solution of the hyperbolic problem in the compressible hodograph plane will also be bounded and the transformed solution will have the same property. However, the solution of a Cauchy problem with highly oscillating initial data for an elliptic equation cannot generally be bounded, because of the presence of highly divergent exponentials. This indicates that something of fundamental qualitative nature must be wrong in the method proposed by Cahn and Garcia.

In fact, it is obvious, by applying the transformation (7) to Eq. (2), that Eq. (1) are not recovered. The resulting system is elliptic as expected from the well-known property that the elliptic, hyperbolic, parabolic, or mixed character of a differential system is conserved by a transformation of independent variables. The transformation is thus meaningless and the paradox is resolved. The arguments proposed by Cahn and Garcia to justify the replacement of a well defined mixed type differential system by a completely different elliptic system are interesting and their fallacy can be shown only by subtler considerations.

Essentially, the algebra involved in deriving Eqs. (4–7) cover a basic assumption which is justified at the end of the section "Outline of Theory." The authors are aware of the

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* Director; also, Professor of Applied Aerodynamics at the University of Liège, Belgium.

fact that the derivative of density with respect to velocity, which occurs in the function g of Eq. (1), is not correctly represented. They accept this incorrect representation on the grounds that this derivative does not occur explicitly in the basic physical equations of the problem, and probably mean, thereby, the integral version of these equations. They conclude from this, that a wildly oscillating representation of $d\rho/dV_c$ may be used, as long as its integral, ρ, approximates the correct $\rho(V_c)$. This could probably be justified by consideration of the integral form of the equations of motion, and we will accept it, although mathematicians might object to violent tinkering with coefficients of partial differential equations.

However, the application that Cahn and Garcia make of this principle is completely erroneous. In fact, they push things too far when they consider a piecewise continuous representation of density in function of velocity as shown on their Fig. 3, which leads to infinite values for the density derivative at the points of discontinuity. This is an essential fact which the authors appear to have missed. If one considers the piecewise continuous representation of Fig. 3 as the limit of a smooth curve with very large slope in small regions around the points of discontinuity, one is led to conclude from the second Eq. (1), that $\partial\phi_c/\partial V_c$ will be of a similar large order of magnitude. In the limiting case of a discontinuous density, one should therefore expect finite jumps in the potential. These jumps are not considered by the authors and their justification for the use of the transformation (7) is therefore not valid, as expected from our earlier arguments.

In the last paragraph of their section "Outline of Theory," Cahn and Garcia present another justification for their assumption about the validity of replacing the actual density by an approximation with wildly oscillating derivative. Although we do not reject this assumption and have shown that the error lies elsewhere, we cannot accept this justification, based on the consideration of the electric analogy tank with variable depth. The equation solved by the analogy is

$$(\partial/\partial x)(h\,\partial V/\partial x) + (\partial/\partial y)(h\,\partial V/\partial y) = 0 \qquad (1)$$

where V is the electric potential and h the depth, a function of the horizontal coordinates x,y. This function must, however, be slowly varying for (1) to be valid. If one considers, with Cahn and Garcia, a sawtooth shaped bottom, for which the depth would contain a wildly oscillating contribution, then a thin layer of three dimensional potential field will develop near the bottom. Outside this layer, a nearly two dimensional potential field will exist, satisfying the equation

$$(\partial/\partial x)(\bar{h}\,\partial V/\partial x) + (\partial/\partial y)(\bar{h}\,\partial V/\partial y) = 0 \qquad (2)$$

where $\bar{h}$ is a smoothed depth distribution involving only slowly varying components. These results can be derived by analyzing the solution of the three-dimensional Laplace equation

$$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0 \qquad (3)$$

with boundary conditions representing the wavy bottom.

Cahn and Garcia appear to believe that the analogy will yield a solution of (1) with wildly oscillating h, but this is incorrect. Their last argument therefore is not relevant to the discussion of their basic assumption.

References

[1] Cahn, M. S. and Garcia, J. R., "Transonic Airfoil Design," *Journal of Aircraft*, Vol. 8 No. 2, Feb. 1971, pp. 84–88.

Reply by Authors to J. Smolderen

M. S. Cahn* and J. R. Garcia†
Northrop Corporation, Hawthorne, Calif.

OUR program has proven very useful for the design of transonic airfoils. Using our method airfoils can be designed to give desired characteristics at transonic speeds, and experiments have shown the resulting design to be sufficiently useful for engineering application. In addition, the design process using our method is extremely rapid; the program runs in only a few seconds on IBM 360.

It would be academically interesting to know exactly what approximations, if any, are involved in our computing process. We therefore would like to thank J. Smolderen for his comments. We feel his observations will be very helpful to our eventual complete understanding of the transonic mixed flow problem.

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* Aircraft Div.
† Aircraft Div.